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SYMPLECTIC 2-HANDLES AND TRANSVERSE LINKS

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Abstract. A standard convexity condition on the boundary of a symplectic manifold involves an induced positive contact form (and contact structure) on the boundary; the corresponding concavity condition involves an induced negative contact form. We present two methods of symplectically attaching 2-handles to convex boundaries of symplectic 4-manifolds along links transverse to the induced contact structures. One method results in concave boundaries and depends on a fibration of the link complement over S^1; in this case the handles can be attached with any framing *larger than* a lower bound determined by the fibration. The other method results in a weaker convexity condition on the new boundary (sufficient to imply tightness of the new contact structure), and in this case the handles can be attached with any framing *less than* a certain upper bound. These methods supplement methods developed by Weinstein and Eliashberg for attaching symplectic 2-handles along Legendrian knots.

1. Results and Motivation

When constructing symplectic manifolds it is natural to wonder whether topological techniques using handles can be made to work symplectically. Weinstein [6] and Eliashberg [2] have shown how to do this in certain cases; here we present two new symplectic "handle-by-handle" constructions in dimension four.

In such constructions it is desirable to retain control of the symplectic form near the boundary; one form of control is the following: Given a symplectic manifold (X, ω) we say that ∂X is *convex* (respectively *concave*) if there exists a vector field V defined in a neighborhood of ∂X, satisfying the equation $\mathcal{L}_V \omega = \omega$ (in other words, V is a symplectic dilation) and pointing *out of* (respectively *into*) X. This induces a contact form $\alpha = \imath_V \omega|_{\partial X}$ and a contact structure $\xi = \ker \alpha$ on ∂X. Weinstein and Eliashberg show that, if (X, ω) is a symplectic $2n$-manifold with ∂X convex, then one can attach k-handles to X, for $0 \leq k \leq n$, and extend ω across the handles so that the new boundary is again convex. Conditions are imposed on the attaching spheres in relation to the contact structure ξ on ∂X and in particular, in dimension four, 2-handles must be attached along *Legendrian* knots (knots tangent to ξ). In this paper we show how to symplectically attach 2-handles along *transverse* knots (transverse to ξ) in the convex boundary of a symplectic 4-manifold so that the new boundary becomes *concave*. Along the way, we see boundaries which are partially convex and partially concave, so we develop a careful theory for such boundaries.

For a weaker form of control, we say that ∂X is *weakly convex* if ∂X supports a positive contact structure ξ such that $\omega|_\xi$ is nondegenerate; convexity implies weak convexity. In this paper we also show how to symplectically attach 2-handles along

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transverse knots in the convex boundary of a symplectic 4-manifold so that the new boundary becomes weakly convex.

1.1. **Theorems and Definitions.** We first present the statements of two theorems followed by the definitions needed to understand the statements. Let (X, ω) be a compact symplectic 4-manifold, suppose that ∂X is convex with induced contact structure ξ and let L be a transverse link in $(\partial X, \xi)$ with a chosen framing F.

Theorem 1.1. *If L is* fat with respect to F *(a condition which holds for negative enough framings), then there exists a symplectic 4-manifold (Y, η) containing (X, ω), obtained by* enlarging (X, ω) *in a neighborhood of L and then attaching 2-handles along L with framing F, such that ∂Y is weakly convex.*

Theorem 1.2. *If L is* nicely fibered *and if F is* positive with respect to the fibration *(a condition which holds for positive enough framings), then there exists a symplectic 4-manifold (Y, η) containing (X, ω), obtained by* enlarging (X, ω) *and then attaching 2-handles along L with framing F, such that ∂Y is concave.*

(For both theorems, more precise statements will be made later about the resulting contact structures on ∂Y.) We will now define "fat with respect to a framing", "nicely fibered" and "positive with respect to a fibration" and we will say what we mean by "enlarging (X, ω)".

First we establish orientation conventions: In this paper all manifolds are assumed to be oriented, and all symplectic forms are assumed to be compatible with the given orientations. A vector field V in a manifold X transverse to a codimension one submanifold M is *positively* transverse if $V(p)$ followed by an oriented basis for T_pM is an oriented basis for T_pX. Boundaries are oriented so that outward normals are positively transverse. Noncritical level sets of a Morse function are oriented so that upward pointing vectors are positively transverse. When M is an oriented manifold, $-M$ will refer to M with the opposite orientation. Unless stated otherwise, all diffeomorphisms and embeddings preserve orientations.

The first definitions involve local control in a neighborhood of a transverse knot K. For this it is convenient to use "polar coordinates" near K, by which we mean functions (r, μ, λ) defining an embedding of a neighborhood ν of K into $\mathbb{R}^2 \times S^1$, with (r, μ) mapping to polar coordinates on $\mathbb{R}^2$ and λ mapping to S^1, and such that $K = \{r = 0\}$. Note that the function $\mu : \nu \setminus K \to S^1$ defines a framing of K, which we label F_μ. If K is transverse to a contact structure ξ, then we can always find *normal* polar coordinates on a neighborhood ν, by which we mean polar coordinates (r, μ, λ) such that $\xi|_{\nu \setminus K}$ is spanned by ∂_r and $\pm \log(r^2)\partial_\mu + \partial_\lambda$. (Where $\pm$ means $+$ when ξ is positive and $-$ when ξ is negative.) This follows from Darboux's theorem for contact structures. (See, for example, [5].) A consequence of this parametrization of r is that, for each integer n, the characteristic foliation on the torus $\{r = e^n\}$ is a family of parallel longitudes realizing the framing $F_\mu + n$. Otherwise, the exact parametrization of r is not particularly important and so we say that (r, μ, λ) is an *almost normal* coordinate system if there exists a function R such that $(R \circ r, \mu, \lambda)$ is a normal coordinate system. (If (r, μ, λ) is an almost normal coordinate system then so is $(r, \mu - k\lambda, \lambda)$ for any $k \in \mathbb{Z}$ and the relation between the framings is that $F_{\mu - k\lambda} = F_\mu + k$.)

Definition 1.3. Given a knot K in a contact 3-manifold (M, ξ) with a framing F and with a chosen neighborhood ν with normal coordinates (r, μ, λ), we say that

the coordinate system *goes out as far as* F if the image of ν in $\mathbb{R}^2 \times S^1$ contains the solid torus $\{r \leq e^n\}$, where $F = F_\mu + n$. Given a link L in (M, ξ) and a framing F of L, with F_i being the framing induced on each component K_i of L, we say that L is *fat with respect to* F if there exist mutually disjoint neighborhoods ν_i of each K_i, each with normal coordinates which go out as far as F_i.

A link L in a 3-manifold M is fibered if $M \setminus L$ fibers over S^1, and one often requires some controlled behavior of the fibration near L. For each fiber Σ, a contact structure ξ on M induces a (singular) *characteristic foliation* on Σ, the integral curves of $T\Sigma \cap \xi$.

A *contact vector field* is a vector field which preserves a given contact structure; we will say that a contact vector field is *transverse* if it is everywhere transverse to the contact plane field. We will be interested in transverse contact vector fields which are also transverse to the fibers of a fibration.

Such a vector field V co-orients both the fibers and the plane field and hence orients the characteristic foliation on each fiber according to the following convention: At a point in M where ξ is transverse to a fiber, let α be a 1-form with kernel ξ such that $\alpha(V) > 0$ and let β be a 1-form with kernel tangent to the fiber such that $\beta(V) > 0$. Then let γ be a 1-form such that $\alpha \wedge \beta \wedge \gamma > 0$. Then $\ker \alpha \cap \ker \beta$ is transverse to $\ker \gamma$ so we restrict γ to define the orientation on the characteristic foliation. This orientation does not change if we replace V with $-V$.

Definition 1.4. A transverse link $L = K_1 \cup \ldots \cup K_n \subset (M, \xi)$ is *nicely fibered* if there exists a fibration $p : M \setminus L \to S^1$ and a transverse contact vector field V (defined on all of M) satisfying the following conditions:

- V is transverse to the fibers of p.
- For each K_i there exist almost normal polar coordinates (r, μ, λ) on a neighborhood ν_i near K_i such that, on ν_i, ∂_r is tangent to the fibers, $dr(V) = 0$ and V and dp are both invariant under the flows of ∂_r, ∂_μ and ∂_λ.
- Letting V co-orient both ξ and the fibers, the oriented characteristic foliation on the fibers near each K_i should point in towards K_i. (By the previous condition the unoriented foliation will be by radial lines in each normal coordinate system.)

Such a fibration has a bearing on framings of L. For each K_i with normal polar coordinates (r, μ, λ) as in the definition, consider a torus $T = \{r = R\}$ with coordinates (μ, λ). The fibers of p intersect T in a family of parallel lines with "slope" $\frac{d\mu}{d\lambda} = s_p \in \mathbb{R} \cup \{\infty\}$. A framing F of L also gives a family of parallel longitudes in T with "slope" $\frac{d\mu}{d\lambda} = s_F \in \mathbb{Z}$. (In terms of the earlier notation, $F = F_\mu + s_F$.)

Definition 1.5. In this situation we say that F is *positive with respect to the fibration* if, for each K_i as above, $s_p \neq \infty$ and $s_F > s_p$.

Definition 1.6. By *enlarging* a symplectic manifold (X, ω) we will mean the following process: Choose a smooth function $h : \partial X \to [0, \infty)$ and let $C = \{(t, p) \mid 0 \leq t \leq h(p)\} \subset \mathbb{R} \times \partial X$. Then attach C to X by the obvious identification of $\{0\} \times \partial X \subset C$ with $\partial X \subset X$. Given a neighborhood N in ∂X, we will say that the enlargement is *supported inside* N, or simply that we are *enlarging* (X, ω) *inside* N, if the support of h is contained in N. After attaching C to X choose some extension of ω to a symplectic form on all of $X \cup C$ and replace (X, ω) with the

new, larger symplectic manifold. There is a natural identification between the old $\partial X = \{0\} \times X$ and the new $\partial X = \{h(p), p\} \in C$.

When ∂X is convex there is a canonical extension of ω over C, constructed as follows: Given a contact form α on a 3-manifold M there is a canonical symplectification $(\mathbb{R} \times M, d(e^t\alpha))$, in which ∂_t is a symplectic dilation inducing α on $\{0\} \times M$. If M is embedded in another symplectic manifold (Y, η) with a symplectic dilation V positively transverse to M inducing α, then flow along V generates a symplectomorphism from a neighborhood of $\{0\} \times M$ in $(\mathbb{R} \times M, d(e^t\alpha))$ to a neighborhood of M in (Y, η), restricting to the identity on M and sending ∂_t to V. Thus if α is the induced contact form on ∂X and we use the form $d(e^t\alpha)$ on $C \subset \mathbb{R} \times \partial X$, this will patch together smoothly with ω on X and the symplectic dilation will extend across C to be transverse to the new ∂X. Furthermore the induced form on the new ∂X is $e^h\alpha$, and so the underlying contact structure is unchanged. For future reference, we will call this enlargement the convex enlargement of height h; it is clearly useful if we would like to rescale the contact form on ∂X by some function greater than or equal to 1.

This is the enlargement used in the two theorems, although in theorem 1.2 we will need to see the enlargement in a more general setting. Enlarging (X, ω) does not change the diffeomorphism type of X and so can instead be thought of as a deformation of ω keeping X fixed.

1.2. **Discussion.** If L is fat with respect to a particular framing then it is also fat with respect to any more negative framing, so theorem 1.1 gives a construction that works for very negative framings but is less likely to work the more positive the framings become. If F is positive with respect to a nice fibration then any more positive framing is also positive with respect to the fibration, so that theorem 1.2 gives a construction that works for very positive framings (assuming the fibration exists) but is less likely to work the more negative the framings become.

It is interesting to see these constructions alongside the construction of Weinstein [6] mentioned earlier. (We use Weinstein as our source because Weinstein's discussion is strictly symplectic whereas Eliashberg [2] discusses the construction in the case of Stein manifolds.) To simplify matters we present the result only in dimension four.

Theorem 1.7 (Weinstein)**.** *Let (X, ω) be a symplectic 4-manifold with ∂X convex with induced contact structure ξ. Then, given any Legendrian knot $K \subset (\partial X, \xi)$ there exists a symplectic 4-manifold (Y, η) containing (X, ω), with ∂Y convex, obtained by enlarging (X, ω) in a neighborhood of K and then attaching a 2-handle along K with framing* $\mathrm{tb}(K) - 1$ *(where* $\mathrm{tb}(K)$ *is the Thurston-Bennequin framing of K). We can also symplectically attach any number of 1-handles to (X, ω) to get (Y, η) with convex boundary.*

(The Thurston-Bennequin framing of K is the framing given by any vector field transverse to K but lying in ξ.) Since any knot is C^0-close to a Legendrian knot and every Legendrian knot can be isotoped so as to make its Thurston-Bennequin framing more negative (see [3]), the 2-handle part of this theorem is also a construction which, given a smooth knot type, works for very negative framings but is less likely to work the more positive the framings become.

This result and the fact that the contact structure on a weakly convex boundary is always tight [1] were used by Gompf [4] to construct many 3-manifolds with tight

contact structures, beginning with the standard positive contact structure on S^3 as the convex boundary of B^4 with its standard symplectic structure. Consider the following observation (suggested by John Etnyre):

Proposition 1.8. *Suppose that K is a Legendrian knot in a positive contact 3-manifold with a given neighborhood ν and a framing $F \leq \mathrm{tb}(K) - 1$. Then there exists a transverse knot K' inside ν, isotopic to K, which is fat (inside ν) with respect to F.*

This tells us that in fact theorem 1.1, together with the 1-handle part of theorem 1.7, can be used to construct the same 3-manifolds that Gompf constructs, also with tight contact structures, but now they are weakly convex rather than convex boundaries of symplectic 4-manifolds.

Constructing manifolds with concave boundaries is interesting for two reasons. First we get some answers to a simple symplectic filling question: which 3-manifolds with which contact structures can be realized as the concave boundaries of symplectic 4-manifolds? Secondly, if we can carefully characterize the contact structures that result from our concave constructions, and also construct symplectic 4-manifolds with convex boundaries and carefully characterize the resulting contact structures, then we may be able to use the following standard glueing construction to produce closed symplectic 4-manifolds: Let (X_1, ω_1) and (X_2, ω_2) be symplectic 4-manifolds with ∂X_1 convex and ∂X_2 concave with induced contact forms α_1 and α_2 and contact structures ξ_1 and ξ_2. Suppose that ∂X_1 and ∂X_2 are connected and that there exists a contactomorphism $\phi : (\partial X_1, \xi_1) \to (-\partial X_2, \xi_2)$. Then $\phi^* \alpha_2 = f \alpha_1$ for some nonzero function f on ∂X_1. Rescaling ω_2 by a constant we can arrange that $f > 1$. Let $h = \log f$. Enlarging (X_1, ω_1) with the canonical convex enlargement of height h will then replace α_1 with $e^h \alpha_1 = f \alpha_1$. Thus we can in fact arrange that $\phi^* \alpha_2 = \alpha_1$. Finally, (X_1, ω_1) can be glued to (X_2, ω_2) by identifying a neighborhood of ∂X_1 with $((-\epsilon, 0] \times \partial X_1, d(e^t \alpha_1))$ and, using ϕ, identifying a neighborhood of ∂X_2 with $([0, \epsilon) \times \partial X_1, d(e^t \alpha_1))$.

For specific examples where theorem 1.2 applies, observe that the transverse unknot and the Hopf link in S^3 with the standard contact structure are both nicely fibered, with framings greater than 0 being positive with respect to the unknot fibration and framings greater than or equal to 0 being positive with respect to the Hopf link fibration. We will see more general examples in section 5.

1.3. **Between convexity and concavity.**

As mentioned earlier, in the process of changing a convex boundary to a concave boundary by attaching handles along a nicely fibered link we encounter boundaries which are partially convex and partially concave. Understanding how to control symplectic forms along such boundaries is essential to the construction.

Definition 1.9. A *dilation-contraction pair* on a symplectic 4-manifold (X, ω) is a pair (V^+, V^-) of vector fields defined respectively on (possibly empty) open subsets X^+ and X^- of X such that the following equations hold (on the sets where they make sense):

$$\mathcal{L}_{V^\pm}(\omega) = \pm\omega, \quad \omega(V^+, V^-) = 0$$

A *contact pair* on a 3-manifold M is a pair (α^+, α^-) of 1-forms, defined respectively on (possibly empty) open subsets $M^\pm$ of M, such that $M = M^+ \cup M^-$ and such

that the following equations hold (on the sets where they make sense):

$$\pm\alpha^\pm \wedge d\alpha^\pm > 0, \quad -d\alpha^- = d\alpha^+$$

If M is an oriented 3-dimensional submanifold of (X, ω) then (V^+, V^-) *transversely covers* M if $M \subset X^+ \cup X^-$ and if each vector field is positively transverse, where defined, to M. Letting $M^\pm = M \cap X^\pm$ and $\alpha^\pm = \imath_{V^\pm}(\omega)|_{M^\pm}$, we see that the *induced pair* (α^+, α^-) is a contact pair on M.

Notice that, for a contact pair (α^+, α^-), α^+ is a positive contact form on M^+ while α^- is a negative contact form on M^-. Together the two 1-forms give a globally defined, closed, nondegenerate 2-form γ such that $\gamma|_{M^\pm} = \pm d\alpha^\pm$.

By a boundary which is partially convex and partially concave, we mean a boundary which is transversely covered by a dilation-contraction pair. (Both convex boundaries and concave boundaries are special cases.) By a germ of a symplectic form along a 3-manifold M we mean an equivalence class of symplectic 4-manifolds containing M, where $(X_1, \omega_1) \sim (X_2, \omega_2)$ if there exist neighborhoods N_i of M in X_i and a symplectomorphism from N_1 to N_2 restricting to the identity on M. We will prove the following result in section 4:

Proposition 1.10. *A contact pair (α^+, α^-) on a 3-manifold M defines a unique symplectic germ $\mathcal{G}(\alpha^+, \alpha^-)$ along M in the following sense:*

1. *There exists a symplectic 4-manifold (X, ω) containing M with a dilation-contraction pair transversely covering M inducing (α^+, α^-).*
2. *Any other symplectic 4-manifold (X_1, ω_1) containing M with the property that $\omega_1|_M = \pm d\alpha^\pm$ represents the same germ along M.*

This in particular implies that the induced contact pair on a partially convex and partially concave boundary uniquely determines the germ of the symplectic form along the boundary; we have already seen this in the purely convex and purely concave cases.

1.4. **Outline.** We will now prove these results in the following order: After establishing some terminology regarding handles, we will prove theorem 1.1 and discuss the relationship with the Legendrian 2-handles of Weinstein. (The constructions of the 2-handles in both theorems will be closely modelled on Weinstein's construction.) Then we will prove the necessary results on partially convex and partially concave boundaries and construct a class of symplectic 2-handles for this setting. Using this more general setup we will prove theorem 1.2 and construct some examples.

For background on basic tools in symplectic and contact constructions, especially various versions of Darboux's theorem for symplectic and contact structures, the reader is referred to [5].

2. Terminology for Handles

Our standard model for an n-dimensional k-handle will be a subset H of $\mathbb{R}^n$ constructed in the following manner: Let f be a Morse function on $\mathbb{R}^n$ with a single critical point of index k at 0, with $f(0) = 0$. Choose constants $\epsilon_1 < 0 < \epsilon_2$; H will be a subset of $f^{-1}[\epsilon_1, \epsilon_2]$ bounded by two smooth codimension one submanifolds with boundary, the "attaching boundary" $\partial_1 H$ and the "free boundary" $\partial_2 H$. See figure 1. The attaching boundary $\partial_1 H$ is a closed tubular neighborhood of the descending sphere K_1 in $f^{-1}\{\epsilon_1\}$, so that $\partial_1 H \cong S^{k-1} \times B^{n-k}$. The free boundary

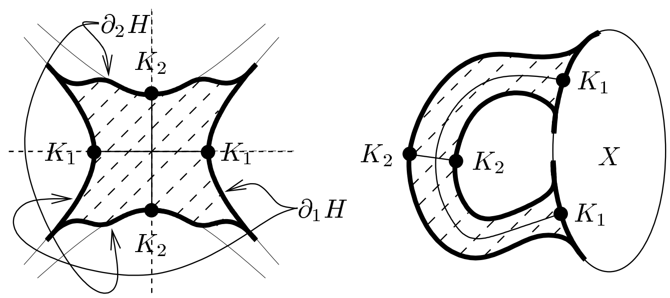

FIGURE 1. Schematic pictures of a handle H and of the result of attaching H to a manifold X

$\partial_2 H$ begins as a tubular neighborhood of the ascending sphere K_2 in $f^{-1}\{\epsilon_2\}$ but then dips down to join $\partial_1 H$ in $f^{-1}\{\epsilon_1\}$ so that $\partial(\partial_1 H) = \partial(\partial_2 H)$. (Thus $\partial_2 H \cong B^k \times S^{n-k-1}$.) Some form of smooth "interpolation" from $f^{-1}\{\epsilon_1\}$ to $f^{-1}\{\epsilon_2\}$ must be specified to construct $\partial_2 H$; in our constructions we will use a vector field transverse to the level sets of f to guide this interpolation.

Our convention is to orient $\partial_1 H$ and $\partial_2 H$ as level sets of f, since H is not technically a manifold with boundary. This gives $\partial_2 H$ the orientation we would expect if it really were a boundary component of H, but gives $\partial_1 H$ the nonstandard orientation; the two orientations agree on $\partial_1 H \cap \partial_2 H$. The descending sphere K_1 comes with a canonical framing in $\partial_1 H$ which we will call the "handle framing" of K_1.

Note in the figure that $\partial_1 H \cap \partial_2 H$ is a codimension one submanifold, diffeomorphic to $S^{k-1} \times S^{n-k-1} \times I$. This "flange" on the handle guarantees the smoothness of the "corners" after attaching H to an n-manifold X. To attach H to X, let $\hat{X}$ be X with an open collar attached to ∂X. We must specify an embedding of an open neighborhood of $\partial_1 H$ in $\mathbb{R}^n$ into $\hat{X}$ restricting to an embedding of $\partial_1 H$ into ∂X. Of course, to do this we actually only need to specify the embedding of $\partial_1 H$ into ∂X but when we add symplectic structures we should be more careful. In fact in the smooth case we need only specify the image K of K_1 in ∂X and the framing F of K induced by the handle framing of K_1 in $\partial_1 H$ to completely determine the diffeomorphism type of the result of attaching H to X.

To construct a symplectic handle (H, ω_0) and attach it symplectically to a symplectic manifold (X, ω) requires a symplectic structure ω_0 on $\mathbb{R}^n$, an extension of ω from X to $\hat{X}$, and a symplectic embedding of a neighborhood of $\partial_1 H$ into $\hat{X}$ restricting to an embedding of $\partial_1 H$ into ∂X. If we have a symplectic dilation positively transverse to $\partial_1 H$ and another symplectic dilation positively transverse to ∂X, then, using the symplectification of the contact forms, we need only specify an embedding of $\partial_1 H$ into ∂X which preserves the induced contact forms in order to specify the symplectic embedding of a neighborhood of $\partial_1 H$. If the embedding of $\partial_1 H$ into ∂X instead only respects the contact structures, then we must enlarge (X, ω) in a neighborhood of the image of $\partial_1 H$ and perhaps rescale ω_0 to arrange that the embedding actually preserves the contact forms.

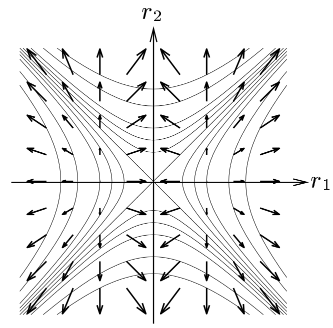

FIGURE 2. Level sets of f and the symplectic dilation V

3. Weakly convex boundaries and proof of theorem 1.1

The idea in the proof of theorem 1.1 is as follows: We use polar coordinates on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$, by which we mean coordinates $(r_1, \theta_1, r_2, \theta_2)$ where (r_i, θ_i) are polar coordinates on the respective $\mathbb{R}^2$ factors. We construct the 4-dimensional 2-handle H as a neighborhood of the origin in $\mathbb{R}^4$, using the Morse function $f = -r_1^2 + r_2^2$. We give $\mathbb{R}^4$ the standard symplectic form and construct a symplectic dilation V which is transverse to the level sets of f wherever it is defined, but which does not extend across $\{r_1 = 0\}$. When we construct H, V will be transverse to both $\partial_1 H$ and $\partial_2 H \setminus K_2$, but will not extend across K_2. Thus V will induce positive contact structures on $\partial_1 H$ and on $\partial_2 H \setminus K_2$. We will show that the contact structure on $\partial_2 H$ can be deformed in a neighborhood of K_2 so that it does extend across K_2, maintaining the non-degeneracy condition needed to get weak convexity. Finally, K_1 will be a transverse knot in $\partial_1 H$ and we will see why a condition must be imposed on the framing in order to attach H along a given transverse knot K.

Proof of theorem 1.1. The standard symplectic form on $\mathbb{R}^4$ is $\omega_0 = r_1 dr_1 d\theta_1 + r_2 dr_2 d\theta_2$. Let $f = -r_1^2 + r_2^2$ and let $V = \frac{1}{2}[(r_1 - \frac{1}{r_1})\partial_{r_1} + r_2 \partial_{r_2}]$. This vector field and some level sets of f are shown in figure 2.

Notice that V is a symplectic dilation and is positively transverse to the level sets of f as long as $f > -1$ and as long as V is defined, but that V does not extend across $\{r_1 = 0\}$. Choose constants ϵ_1 and ϵ_2 with $-1 < \epsilon_1 < 0 < \epsilon_2$. The handle H will be a subset of $f^{-1}[\epsilon_1, \epsilon_2]$, with $\partial_1 H \subset f^{-1}\{\epsilon_1\}$, $K_1 = \{r_2 = 0\} \cap f^{-1}\{\epsilon_1\}$ and $K_2 = \{r_1 = 0\} \cap f^{-1}\{\epsilon_2\}$.

First we calculate the contact forms induced by V on $f^{-1}\{\epsilon_1\}$ and $f^{-1}\{\epsilon_2\} \setminus K_2$. Both are restrictions of the form $\imath_V \omega_0 = \frac{1}{2}[(r_1^2 - 1) d\theta_1 + r_2^2 d\theta_2]$. On $f^{-1}\{\epsilon_1\}$, natural polar coordinates consistent with our orientation convention are $(r = r_2, \mu = \theta_2, \lambda = -\theta_1)$. With respect to these coordinates we get $\alpha_1 = \frac{1}{2}[r^2 d\mu - (r^2 - \epsilon_1 - 1) d\lambda]$. Natural polar coordinates on $f^{-1}\{\epsilon_2\}$ are $(r = r_1, \mu = \theta_1, \lambda = \theta_2)$ and with respect to these coordinates we get $\alpha_2 = \frac{1}{2}[(r^2 - 1) d\mu + (r^2 + \epsilon_2) d\lambda]$.

Now, given any small positive δ, we will show how to modify the contact structure $\xi_2 = \ker \alpha_2$ inside a neighborhood $\{r^2 \leq 1 + \delta\}$ of K_2 to get a new contact structure ξ_2' which extends across K_2 and still satisfies the property that $\omega_0|_{\xi_2'}$ is nondegenerate. Let $\xi_2' = \ker \alpha_2'$ where $\alpha_2' = d\lambda + t(r^2) d\mu$ and $t(r^2)$ goes smoothly

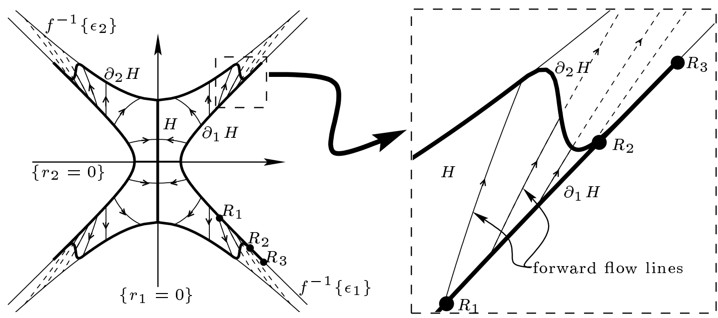

FIGURE 3. Construction of H in the proof of theorem 1.1

to 0 as r^2 goes to 0, has positive derivative and is equal to $\frac{r^2-1}{r^2+\epsilon_2}$ on $\{r^2 \geq 1+\delta\}$. This agrees with ξ_2 outside $\{r^2 \leq 1+\delta\}$ because $\xi_2 = \ker(d\lambda + \frac{r^2-1}{r^2+\epsilon_2}d\mu)$, is contact because $t' > 0$ and satisfies the nondegeneracy condition because $\alpha_2' \wedge (\omega_0|_{f^{-1}\{\epsilon_2\}}) = (1-t(r^2))rdr \wedge d\mu \wedge d\lambda > 0$. Intuitively, ξ_2 twists too far as we move in towards K_2 to extend across K_2, so we back off to $\{r^2 = 1+\delta\}$ and then twist more slowly so that ξ_2' does extend.

Forward flow along V for time t starting at a point $(r,\mu,\lambda) \in f^{-1}\{\epsilon_1\}$ gives a map Φ from a subset of $\mathbb{R} \times f^{-1}\{\epsilon_1\}$ into $\mathbb{R}^4$ defined by the following equations:

$$\begin{aligned} r_1^2 \circ \Phi &= (r^2 - \epsilon_1 - 1)e^t + 1\,, \quad & \theta_1 \circ \Phi &= -\lambda \\ r_2^2 \circ \Phi &= r^2 e^t\,, & \theta_2 \circ \Phi &= \mu \end{aligned}$$

Letting $R = \sqrt{\epsilon_2(\frac{1+\epsilon_1}{1+\epsilon_2})}$ and $T = \log(\frac{1+\epsilon_2}{1+\epsilon_1})$, we see that forward flow for time $t = T$ defines a diffeomorphism $\phi : f^{-1}(\epsilon_1) \setminus \{r \leq R\} \to f^{-1}(\epsilon_2) \setminus K_2$. Note that $1+\epsilon_1 > R^2$ and that $\phi\{r^2 = 1+\epsilon_1\} = \{r^2 = 1\}$.

Given any three radii $R_3 > R_2 > R_1 > \sqrt{1+\epsilon_1}$ we can construct a symplectic handle H as follows: Choose a smooth function $h : [0,R_3] \to [0,T]$ which is equal to T on $[0,R_1]$, is decreasing on $[R_1,R_2]$ and is equal to 0 on $[R_2,R_3]$. Then let H be the union of all the forward flow lines starting at points $p \in \{r \leq R_3\} \subset f^{-1}\{\epsilon_1\}$ flowing for time less than or equal to $h(r(p))$, together with $\{r_1 = 0\} \cap f^{-1}[\epsilon_1,\epsilon_2]$. The attaching boundary $\partial_1 H$ is $\{r \leq R_3\} \subset f^{-1}\{\epsilon_1\}$ while the free boundary $\partial_2 H$ is the image under Φ of the graph of h in $\mathbb{R} \times f^{-1}\{\epsilon_1\}$ together with K_2. This construction is illustrated in figure 3. In the figure R_3, R_2 and R_1 are rather far apart for the sake of clarity, but in general one would carry out this construction with these radii only slightly larger than $\sqrt{1+\epsilon_1}$. A few forward flow lines for V are shown, starting on $f^{-1}\{\epsilon_1\}$.

The construction of ξ_2' on $f^{-1}\{\epsilon_2\}$ gives a positive contact structure ξ_2'' on $\partial_2 H$ by letting $\xi_2'' = \xi_2'$ on $\partial_2 H \cap f^{-1}\{\epsilon_2\}$ and letting $\xi_2'' = \ker(\imath_V \omega_0|_{\partial_2 H})$ elsewhere. The nondegeneracy condition on ξ_2' and the fact that V is positively transverse to $\partial_2 H$ where defined gives us that $\omega_0|_{\xi_2''}$ is nondegenerate. Furthermore ξ_2'' agrees with $\xi_1 = \ker\alpha_1$ on $\partial_1 H \cap \partial_2 H$, so that if we succeed in attaching H to (X,ω) via a contactomorphic embedding of $(\partial_1 H, \xi_1)$ into $(\partial X, \xi)$, the contact structures will patch together to give a contact structure ξ_Y on the boundary ∂Y of the new symplectic manifold (Y,η) with $\eta|_{\xi_Y}$ nondegenerate.

Now we show how to find this embedding and attach H. Given the transverse link $L \subset \partial X$ which is fat with respect to a framing F, consider a component K of L with neighborhood ν with almost normal coordinates (r, μ, λ) which go out as far as F. Notice that we can choose these coordinates so that $F_\mu = F$. This means that the contact structure ξ near K is spanned by ∂_r and $\log(r^2)\partial_\mu + \partial_\lambda$ and that $\{r \leq 1\} \subset \nu$. Choose some $\epsilon > 0$ such that $\{r \leq 1 + \epsilon\} \subset \nu$. Now consider the coordinates (r, μ, λ) on $\partial_1 H$; these are almost normal coordinates and we could explicitly reparametrize r to make them normal. However it is sufficient to notice that ξ_1 is spanned by ∂_r and $\frac{r^2-\epsilon_1-1}{r^2}\partial_\mu + \partial_\lambda$ and that this implies that, after reparametrizing r, the coordinates would go out as far as the handle framing F_μ. This is because we construct H with $R_3 > R_2 > R_1 > \sqrt{1+\epsilon_1}$. This means that if we construct H with $R_3 - \sqrt{1+\epsilon_1}$ small enough we can guarantee a contactomorphism from $\partial_1 H$ to $\{r \leq 1 + \epsilon\} \subset \nu$, taking the handle framing to F. $\square$

An attractive feature of this construction is that it is straightforward to visualize the resulting "contact surgery". The new contact manifold $(\partial Y, \xi_Y)$ is constructed from $(\partial X, \xi)$ as follows: For each component K_i of the transverse link L, let ν_i be a neighborhood of K_i with normal coordinates (r_i, μ_i, λ_i) which go out at least as far as the framing F_i. For each i choose a small positive ϵ_i so that $\{r_i \leq e^{n_i} + \epsilon_i\}$ is in the image of the coordinate system, where $F_i = F_{\mu_i} + n_i$. For each i remove the solid torus $\{r_i < e^{n_i} + \epsilon_i/2\}$ and glue back in a solid torus ν_i' by overlapping along $\{e^{n_i} + \epsilon_i/2 \leq r_i \leq e^{n_i} + \epsilon_i\}$, realizing the (topological) surgery with framing F_i. Then we can extend ξ across ν_i' exactly because, on $\{e^{n_i} + \epsilon_i/2 \leq r_i \leq e^{n_i} + \epsilon_i\}$, ξ is twisting towards the longitude realizing the framing F_i, which, after the surgery, will become a meridian. We needed to remove at least $\{r_i \leq e^{n_i}\}$ because, when $r_i < e^{n_i}$, ξ has already twisted past this longitude.

To see that this construction achieves all the surgeries achievable using Weinstein's Legendrian surgeries, we now present the proof of proposition 1.8, beginning with a lemma.

Lemma 3.1. *If β_1 and β_2 are 1-forms on disks D_1 and D_2 such that $d\beta_i > 0$ then $\alpha_i = d\lambda + \beta_i$ are positive contact forms on $D_i \times S^1$ (λ being the S^1-coordinate). If $\phi : (D_1, d\beta_1) \to (D_2, d\beta_2)$ is a symplectomorphism then there exists a function $h : D_1 \to \mathbb{R}$ such that the diffeomorphism $\Phi : D_1 \times S^1 \to D_2 \times S^1$ given by $\Phi(p, \lambda) = (\phi(p), \lambda + h(p))$ satisfies $\Phi^*\alpha_2 = \alpha_1$.*

Proof. That each α_i is a positive contact form is a straightforward calculation. To see why the rest is true, note that since $\phi^* d\beta_2 = d\beta_1$, the 1-form $\beta_1 - \phi^*\beta_2$ is closed and therefore exact. Choose h so that $\beta_1 = \phi^*\beta_2 + dh$. Then $\Phi^*\alpha_2 = d(\Phi^*\lambda) + \phi^*\beta_2 = d\lambda + dh + \phi^*\beta_2 = \alpha_1$. $\square$

Proof of proposition 1.8. Let K be the Legendrian knot, ν a neighborhood of K and $F \leq \mathrm{tb}(K) - 1$ a framing of K. Without loss of generality, by Darboux's theorem for contact structures, we may assume that ν has the form $(\nu = D_\epsilon \times S^1, \xi = \ker \alpha)$ where $\alpha = dy - x d\lambda$, x and y are coordinates on the disk $D_\epsilon = \{x^2 + y^2 < \epsilon^2\}$ of radius ϵ, and λ is the S^1-coordinate. This is because $K = \{0\} \times S^1$ is Legendrian. Note that in this model $\mathrm{tb}(K)$ is the "zero-framing" coming from the product structure on ν. We will measure framings relative to this product framing, so that $\mathrm{tb}(K) - 1 = -1$.

On $\{x > 0\}$, we have $\xi = \ker \alpha'$, where $\alpha' = d\lambda + \beta$ and $\beta = -\frac{1}{x}dy$. The 2-form $d\beta = \frac{1}{x^2}dx \wedge dy$ is positive on $\{x > 0\}$. We will construct a symplectomorphism ϕ from the disk D_2 of radius 2 in $\mathbb{R}^2$ with the standard symplectic form $dx \wedge dy = rdr \wedge d\mu$ onto a region $D \subset \{x > 0, x^2 + y^2 < \epsilon\}$ with the symplectic form $d\beta$. Then on D_2 let $\beta_2 = \frac{1}{2}r^2 d\mu$ and note that $d\beta_2 = rdr \wedge d\mu$. Thus lemma 3.1 gives a contactomorphism Φ from $(D_2 \times S^1, d\lambda + \beta_2)$ onto $(D \times S^1, \alpha')$ taking the zero framing to the zero framing. Finally note that $K_2 = \{r = 0\} \subset (D_2 \times S^1, d\lambda + \beta_2)$ is fat with respect to the framing $-1 = \mathrm{tb}(K) - 1$.

We construct ϕ directly. Choose two positive constants c_1 and c_2, define ϕ by:

$$\phi(x, y) = (\frac{c_2}{c_1 - x}, c_2 y)$$

and verify that $\phi^* \frac{1}{x^2} dx \wedge dy = dx \wedge dy$. The map is only defined when $x < c_1$, but as long as we choose $c_1 > 2$, ϕ will be defined on D_2. By choosing c_2 small enough we can guarantee that $\phi(D_2) \subset \{x > 0, x^2 + y^2 < \epsilon\} = D$. □

4. Partially convex and partially concave boundaries

Before proving theorem 1.2 we need to develop a theory of symplectic boundaries which are partially convex and partially concave. (We only develop this theory in dimension four.) The definitions were given in section 1; our first task is to show that a contact pair induced by a dilation-contraction pair uniquely determines the germ of the symplectic form and to look at some corollaries of the proof. After that we will show how to construct a symplectic 2-handle H with a dilation-contraction pair transversely covering $\partial_1 H$ and $\partial_2 H$ and inducing specified contact pairs. In the next section we will use these tools to prove theorem 1.2.

4.1. **Uniqueness of germs determined by contact pairs.** Henceforth the notation $(M, (\alpha^+, \alpha^-))$ will refer to a 3-manifold equipped with a contact pair. We will always refer to the domains of the forms as $M^\pm$ and will let $M^0 = M^+ \cap M^-$ with $\alpha^0 = \alpha^+|_{M^0} + \alpha^-|_{M^0}$. A positive (resp. negative) contact form is a special case of a contact pair, with $M^- = \emptyset$ (resp. $M^+ = \emptyset$). Let $R_{\alpha^\pm}$ be the Reeb vector fields for $\alpha^\pm$, and note that α^0 is closed and nowhere zero and that $\alpha^0(R_{\alpha^\pm}) > 1$.

Lemma 4.1. *Given $(M, (\alpha^+, \alpha^-))$, consider the two symplectic manifolds (S^+, ω^+) and (S^-, ω^-), where $S^\pm = \mathbb{R} \times M^\pm$ and $\omega^\pm = \pm d(e^{\pm t}\alpha^\pm)$, and identify $M^\pm$ with $\{0\} \times M^\pm \subset S^\pm$. Firstly, if (X, ω) is another symplectic manifold containing M^+ or M^- with a single symplectic dilation V^+ or contraction V^- positively transverse to $M^\pm$ inducing the contact form α^+ or α^-, respectively, then flow along $V^\pm$ starting from $M^\pm$ gives an embedding Φ of an open subset of S^+ or S^-, respectively, into X such that $\Phi^*\omega = \omega^\pm$ and $D\Phi(\partial_t) = V^\pm$.*

Secondly:

1. *There exists a unique vector field V^- defined on $\mathbb{R} \times M^0$ such that the pair $(V^+ = \partial_t, V^-)$ is a dilation-contraction pair on (S^+, ω^+) inducing the contact pair $(\alpha^+, \alpha^-|_{M^0})$ on M^+.*
2. *There exists a unique vector field V^+ defined on $\mathbb{R} \times M^0$ such that the pair $(V^+, V^- = \partial_t)$ is a dilation-contraction pair on (S^-, ω^-) inducing the contact pair $(\alpha^+|_{M^0}, \alpha^-)$ on M^-.*

We will call the symplectic manifold (S^+, ω^+) with its dilation-contraction pair the *positive symplectification* of $(M^+, (\alpha^+, \alpha^-|_{M^0}))$ and we will call (S^-, ω^-) with its dilation-contraction pair the *negative symplectification* of $(M^-, (\alpha^+|_{M^0}, \alpha^-))$.

Proof. The first result follows from the fact that $\Phi^*\omega$ and $\omega^\pm$ are solutions to the same ordinary differential equations with the same initial conditions.

For the second result, let $\gamma = \pm d\alpha^\pm$, let $g^\pm = \frac{\alpha^0\wedge\gamma}{\alpha^\pm\wedge\gamma} = \alpha^0(R_{\alpha^\pm})$ and let $\beta^\pm = \alpha^0 - g^\pm\alpha^\pm$. We will show that there exist unique vector fields $Z^\pm \in \ker\alpha^+ \cap \ker\alpha^-$ on M^0 such that $\imath_{Z^\pm}(\gamma) = \beta^\pm$. Then the vector fields V^- and V^+, on (S^+,ω^+) and (S^-,ω^-) respectively, are given by the following formulae:

1. $V^- = (g^+e^{-t} - 1)\partial_t + e^{-t}Z^+$
2. $V^+ = (g^-e^{t} - 1)\partial_t + e^{t}Z^-$

We first show the existence and uniqueness of $Z^\pm$, then show that $V^\pm$ as described in these formulae satisfy the conditions of the lemma, and then show uniqueness of $V^\pm$.

There exists a unique $Z^\pm \in \ker\alpha^\pm$ such that $\imath_{Z^\pm}(\gamma) = \beta^\pm$ because contraction with γ gives a linear isomorphism from $\ker\alpha^\pm$ to $\{\beta \mid \beta\wedge\gamma = 0\}$ (this depends on working in dimension 3), and $\beta^\pm$ is constructed to be in this latter subspace. But $Z^\pm$ is also in $\ker\alpha^0$ because $0 = \gamma(Z^\pm, Z^\pm) = \beta^\pm(Z^\pm) = \alpha^0(Z^\pm)$, and thus $Z^\pm \in \ker\alpha^\mp$.

On S^+, letting $V^+ = \partial_t$ and $V^- = (g^+e^{-t}-1)\partial_t + e^{-t}Z^+$, we need to show that

$$\mathcal{L}_{V^\pm}(\omega^+) = \pm\omega^+ \tag{4.1}$$

$$\imath_{V^\pm}(\omega^+)|_{t=0} = \alpha^\pm \tag{4.2}$$

$$\omega^+(V^+, V^-) = 0 \tag{4.3}$$

First note that $\omega^+ = e^t(dt\wedge\alpha^+ + \gamma)$. Equation 4.3 is quick: $\omega^+(V^+,V^-) = e^{-t}\omega^+(\partial_t, Z^+) = \alpha^+(Z^+) = 0$. To show equation 4.1 and equation 4.2, note that $\imath_{\partial_t}(\omega^+) = e^t\alpha^+$ and that

$$\begin{aligned}
\imath_{V^-}(\omega^+) &= (g^+e^{-t}-1)\imath_{\partial_t}(\omega^+) + e^{-t}\imath_{Z^+}(\omega^+) \\
&= e^t[(g^+e^{-t}-1)\alpha^+ + e^{-t}\beta^+] \\
&= -e^t\alpha^+ + \alpha^0
\end{aligned}$$

Next we will prove that V^- is the unique vector field on $\mathbb{R}\times M^0$ satisfying these equations. Suppose V_0^- and V_1^- are two solutions. Let $\delta = \imath_{V_1^- - V_0^-}(\omega^+)$; we will prove that $\delta = 0$ and thus conclude that $V_0^- = V_1^-$. Equation 4.1 implies that $d\delta = 0$, equation 4.2 implies that $\delta|_{\{t=0\}} = 0$ and equation 4.3 implies that $\delta(\partial_t) = 0$ everywhere. Thus δ is invariant in the t direction and vanishes when $t = 0$, so $\delta = 0$ everywhere.

On S^- the argument is a mirror image of the argument for S^+. □

Note that the uniqueness argument first proved uniqueness *along* M (since $\delta|_{\{t=0\}} = 0$) and then proved uniqueness for all t. Thus in fact we have also proved:

Lemma 4.2. *In $(\mathbb{R}\times M^0, \omega^+)$ there exists a unique vector field V^- along M^0 (i.e. a section of $T_{M^0}(\mathbb{R}\times M^0)$) positively transverse to M^0 such that $\imath_{V^-}\omega^+ = \alpha^-$ and $\omega^+(\partial_t, V^-) = 0$. Likewise there exists a unique vector field V^+ along M^0 in $(\mathbb{R}\times M^0, \omega^-)$ positively transverse to M^0 such that $\imath_{V^+}\omega^- = \alpha^+$ and $\omega^-(V^+, \partial_t) = 0$.*

We will make use of the following observation in later constructions:

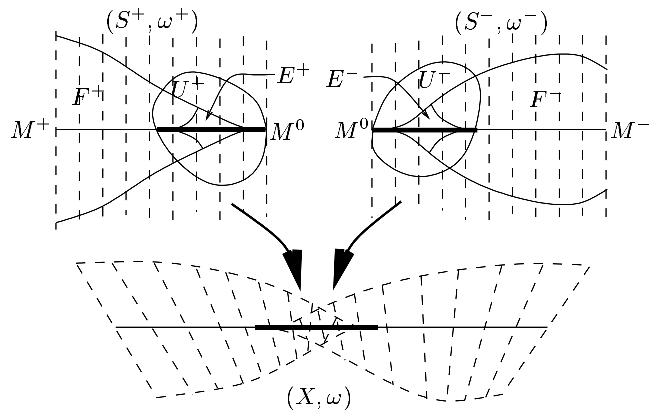

FIGURE 4. Glueing S^+ to S^- after some trimming

Lemma 4.3. *Given $(M,(\alpha^+,\alpha^-))$, let $(S^\pm,\omega^\pm)$ be the positive and negative symplectifications with the dilation-contraction pairs from lemma 4.1. Let $g^\pm$, $\beta^\pm$ and $Z^\pm$ be as in the proof of lemma 4.1. Given a function $h : M^\pm \to \mathbb{R}$, consider its graph $M_h^\pm = \{(h(p),p)\} \subset S^\pm$. Then $V^\pm = \partial_t$ is automatically positively transverse to $M_h^\pm$ and $V^\mp$ is positively transverse to $M_h^\pm$ if and only if $e^{\pm h} < g^\pm - dh(Z^\pm)$ on M^0. Identifying M_h with M in the obvious way, the induced contact pair on M_h is then given by*

$$\alpha_h^\pm = e^h\alpha^\pm,\ \alpha_h^0 = \alpha^0,\ \alpha_h^\mp = \alpha^0 - e^h\alpha^\pm\ .$$

Proof. Everything follows from the explicit expressions $V^\mp = (g^\pm e^{\mp t} - 1)\partial_t + e^{\mp t}Z^\pm$ and $\omega^\pm = e^{\pm t}(dt\wedge\alpha^\pm + \gamma)$. □

Proof of Proposition 1.10. First we prove existence of (X,ω) with its dilation-contraction pair (V^+,V^-). Construct $(S^\pm,\omega^\pm)$ as in lemma 4.1. Let U^+ be an open neighborhood of M^0 in S^+ such that flow along V^+ in S^- starting from $p_0 \in M^0$ is defined for all times t with $(t,p_0) \in U^+$. This gives an embedding $\phi^+ : U^+ \hookrightarrow S^-$ such that $\phi^+|_{M^0} = \mathrm{id}$ and $D\phi^+(\partial_t) = V^+$. Since both ∂_t and V^+ induce the same contact forms on M^0 and are both symplectic dilations, we can conclude that $(\phi^+)^*(\omega^-) = \omega^+$. By the uniqueness in lemma 4.1 we also know that $D\phi^+(V^-) = \partial_t$, so that ϕ^+ preserves all the relevant structure. Let $U^- = \phi^+(U^+)$ and $\phi^- = (\phi^+)^{-1}$.

Now choose two functions $f^\pm : M \to [0,\infty)$ such that $f^\pm|_{M^\mp\setminus M^0} = 0$ but $f^+ + f^- > 0$ everywhere, let $F^\pm = \{(t,p) \mid -f^\pm(p) < t < f^\pm(p)\} \subset S^\pm$ and let $E^\pm = F^\pm \cap \phi^\mp(F^\mp \cap U^\mp)$. Finally let $\psi^\pm = \phi^\pm|_{E^\pm} : E^\pm \to E^\mp$. If we choose $f^\pm$ small enough we can guarantee that

$$X = F^+ \cup_{\psi^+} F^-$$

is Hausdorff. Since $(\psi^+)^*\omega^- = \omega^+$ and $D\psi^+(V^\pm) = V^\pm$, we know that the symplectic forms and the dilation-contraction pairs patch together to define a symplectic form ω on X with a dilation-contraction pair (V^+,V^-) transversely covering M inducing (α^+,α^-). (See figure 4.)

For the uniqueness result, we need to construct a bundle isomorphism $\Psi : T_M X_1 \to T_M X$ covering the identity and preserving the symplectic forms, and then we can apply Darboux's theorem. To do this we construct a pair of vector fields (V_1^+, V_1^-) along M in $T_M X_1$ with open domains $M_1^\pm \subset M^\pm$ covering M, both positively transverse to M, such that $\imath_{V_1^\pm}\omega_1|_{M_1^\pm} = \alpha^\pm|_{M_1^\pm}$ and such that $\omega_1(V_1^+, V_1^-) = 0$. Then there exists a unique Ψ sending $V_1^\pm$ to $V^\pm|_{M_1^\pm}$ by lemma 4.2. To see the existence of the pair (V_1^+, V_1^-), first extend α^+ to a maximal rank 1-form on $T_{M^+}X_1$ to get V_0^+ along M^+ such that $\imath_{V_0^+}\omega_1|_{M^+} = \alpha^+$. Then by lemma 4.2 there exists a unique V_0^- along M^0 such that $\imath_{V_0^-}\omega_1|_{M^0} = \alpha^-|_{M^0}$ and $\omega_1(V_0^+, V_0^-) = 0$. Let $\tilde{\alpha}_0^- = \imath_{V_0^-}\omega_1$, a maximal rank 1-form on $T_{M^0}X_1$ which extends $\alpha^-|_{M^0}$. Then there exists a maximal rank extension $\tilde{\alpha}_1^-$ of α^- to $T_{M^-}X_1$ which agrees with $\tilde{\alpha}_0^-$ outside a closed set C inside M^- containing $M^- \setminus M^0$ in its interior. Let V_1^- be the corresponding vector field such that $\imath_{V_1^-}\omega_1 = \tilde{\alpha}_1^-$ and let $V_1^+ = V_0^+|_{M^+\setminus C}$.

□

Corollary 4.4. *Suppose that (X, ω) is a symplectic 4-manifold with ∂X partially convex and partially concave, with induced contact pair (α_1^+, α_1^-). Notice that $\omega|_{\partial X} = \pm d\alpha_1^\pm$. Now suppose that (α_2^+, α_2^-) is another contact pair on ∂X with $\omega|_{\partial X} = \pm d\alpha_2^\pm$. Then $\mathcal{G}(\alpha_1^+, \alpha_1^-) = \mathcal{G}(\alpha_2^+, \alpha_2^-)$, so that ∂X can also be seen as partially convex and partially concave with induced contact pair (α_2^+, α_2^-). This holds in particular if (α_2^+, α_2^-) is obtained from (α_1^+, α_1^-) by extending or restricting the domains of the 1-forms maintaining the contact pair properties.*

4.2. **Construction of a handle.** Now we will show how to construct a symplectic 2-handle H according to the following plan: As in section 3, H will be a subset of $f^{-1}[\epsilon_1, \epsilon_2] \subset (\mathbb{R}^4, \omega_0)$ for the Morse function $f = -r_1^2 + r_2^2$, where $\epsilon_1 < 0 < \epsilon_2$. In this case, however, we will consider a particular dilation-contraction pair (V^+, V^-) on $\mathbb{R}^4$ with V^+ defined on $\mathbb{R}^4 \setminus \{r_1 = 0\}$ and V^- on $\mathbb{R}^4 \setminus \{r_2 = 0\}$. This pair will transversely cover both $\partial_1 H$ and $\partial_2 H$, inducing contact pairs (α_1^+, α_1^-) and (α_2^+, α_2^-). Let $\partial_i^\pm H$ be the respective domains of these 1-forms; we then have $\partial_1^+ H = \partial_1 H$, $\partial_1^- H = \partial_1 H \setminus K_1$, $\partial_2^+ H = \partial_2 H \setminus K_2$ and $\partial_2^- H = \partial_2 H$. We will use the flow along V^+ to guide the construction of $\partial_2 H$. The tricky point is to arrange that *both* vector fields end up transverse to $\partial_2 H$. For this we will use lemma 4.3 and choose our "interpolation" function h carefully.

We use the same coordinates on $f^{-1}\{\epsilon_1\}$ and $f^{-1}\{\epsilon_2\}$ that we used in section 3: On $f^{-1}\{\epsilon_1\}$, these are $(r = r_2, \mu = \theta_2, \lambda = -\theta_1)$ while on $f^{-1}\{\epsilon_2\}$ these are $(r = r_1, \mu = \theta_1, \lambda = \theta_2)$.

To characterize the contact pairs on $\partial_1 H$ we need a little more terminology. Let ν be a neighborhood of a knot K with a contact pair (α^+, α^-) and with polar coordinates (r, μ, λ). Let the domain of $\alpha^\pm$ be $\nu^\pm$, assume that $\nu^+ = \nu$ and $\nu^- = \nu \setminus K$ and that (r, μ, λ) is an almost normal coordinate system for $\ker \alpha^+$. Let $\alpha^0 = \alpha^+ + \alpha^-$ (in which case the pair (α^+, α^0) determines α^-).

Definition 4.5. Such a contact pair (α^+, α^-) is *well-behaved* with respect to (r, μ, λ) if $R_{\alpha^+} = A\partial_\mu + B\partial_\lambda$ and $\alpha^0 = C d\mu + D d\lambda$ for constants A, B, C, D with B and C positive. The contact pair is *prepared for surgery* with respect to (r, μ, λ) if we also

have that $A = B$, $D > 0$ and $AD > 1$. In each case, we may also say that the knot K is well-behaved or prepared for surgery.

The fact that $\alpha^0(R_{\alpha^+}) > 1$ implies that $AC+BD > 1$. We will call the quadruple (A, B, C, D) the *structural data* for the contact pair (with respect to the coordinate system); this data completely determines (α^+, α^-) on ν up to a reparametrization of r. A convenient model, given (A, B, C, D), is $\alpha^+ = \frac{1}{B+Ar^2}(r^2 d\mu + d\lambda)$ and $\alpha^- = \alpha^0 - \alpha^+ = Cd\mu + Dd\lambda - \alpha^+$. (Simply verify that this is a well-behaved contact pair with the given data.)

Proposition 4.6. *Given a contact pair (α^+, α^-) on a neighborhood ν of a knot K which is prepared for surgery with structural data $(A = B, C, D)$, there exists a handle H such that the induced pair (α_1^+, α_1^-) on $\partial_1 H$ is prepared for surgery with the same structural data (with respect to the above coordinates on $\partial_1 H \subset f^{-1}\{\epsilon_1\}$). Furthermore, with the contact pair on $f^{-1}\{\epsilon_1\}$ fixed, we can construct H so as to make $\partial_1 H$ arbitrarily small as a neighborhood of K_1.*

Thus, given any symplectic 4-manifold X with boundary which is partially convex and partially concave with induced contact pair (α^+, α^-), we can attach such a handle to X whenever we can find a knot K in $(\partial X, (\alpha^+, \alpha^-))$ which is prepared for surgery with respect to some coordinate system. The coordinate system determines the framing, and the handle can be attached along an arbitrarily small neighborhood of K. This is in contrast to the construction in section 3, which required the attaching neighborhood to be "fat enough" with respect to the chosen framing.

Proof. Let $\omega_0 = r_1 dr_1 d\theta_1 + r_2 dr_2 d\theta_2$ on $\mathbb{R}^4$ with $f = -r_1^2 + r_2^2$, and let

$$V^+ = (\frac{1}{2}r_1 - \frac{C}{r_1})\partial_{r_1} + \frac{1}{2}r_2\partial_{r_2}$$
$$V^- = -\frac{1}{2}r_1\partial_{r_1} - (\frac{1}{2}r_2 - \frac{D}{r_2})\partial_{r_2}$$

Calculation shows that (V^+, V^-) is a dilation-contraction pair which transversely covers the level sets of f as long as $-2D < f < 2C$. Let $\epsilon_1 = \frac{2}{A} - 2D$ and note that $-2D < \epsilon_1 < 0$ (because $A > 0$ and $AD > 1$). Choose any ϵ_2 with $0 < \epsilon_2 < 2C$. Then the induced contact pair on $f^{-1}(\epsilon_1)$ is given by:

$$\alpha_1^+ = \frac{1}{2}[r^2(d\mu - d\lambda)] + \frac{1}{A}d\lambda, \quad \alpha_1^0 = Cd\mu + Dd\lambda,$$

with

$$R_{\alpha_1^+} = A(\partial_\mu + \partial_\lambda)\ .$$

Forward flow along V^+ gives a map Φ from some subset of $\mathbb{R} \times f^{-1}(\epsilon_1)$ into $\mathbb{R}^4$ defined by:

$$\begin{aligned} r_1^2 \circ \Phi &= (r^2 - \tfrac{2}{A})e^t + 2D\ , \quad & \theta_1 \circ \Phi &= -\lambda \\ r_2^2 \circ \Phi &= r^2 e^t\ , & \theta_2 \circ \Phi &= \mu \end{aligned}$$

Letting $R = \frac{\epsilon_2}{A(D+\epsilon_2/2)}$ and $T = \log(A(D+\frac{\epsilon_2}{2}))$ (which is positive because $AD > 1$), we see that forward flow for time $t = T$ defines a diffeomorphism $\phi : f^{-1}\{\epsilon_1\} \setminus \{r^2 \leq R\} \to f^{-1}\{\epsilon_2\} \setminus K_2$. We can make R arbitrarily small by choosing ϵ_2 small enough. Now for any choice of radii $R_3 > R_2 > R_1 > R$ we can build H exactly as in section 3, using a function $h : [0, R_3] \to [0, T]$ which is T on $[0, R_1]$, decreasing on $[R_1, R_2]$ and 0 on $[R_2, R_3]$.

By construction $\partial_2 H$ is transverse to V^+. The only part of $\partial_2 H$ which could fail to be transverse to V^- is $\Gamma_h = \Phi(\{(h(r(p)), p) \mid R_1 \leq r(p) \leq R_2\})$. Using lemma 4.3 we can state conditions for Γ_h to be transverse to V^-. Using the notation from the proof of lemma 4.1 we get:

$$\begin{aligned}\gamma &= rdr \wedge (d\mu - d\lambda)\\ g^+ &= A(C+D)\\ \beta^+ &= (C - A(C+D)r^2)(d\mu - d\lambda)\\ Z^+ &= \frac{1}{r}(C - A(C+D)r^2)\partial_r\end{aligned}$$

Thus h needs to satisfy

$$e^{h(r)} < g^+ - h'(r)dr(Z^+) = A(C+D) - \frac{1}{r}h'(r)(C - A(C+D)r^2)$$

for all $r \in [R_1, R_2]$. Since we will have $h' \leq 0$, $e^h \leq A(D + \frac{\epsilon_2}{2})$ and $\epsilon_2 < 2C$, this will work as long as

$$r^2 < \frac{C}{A(C+D)} .$$

Therefore if we build H with $R_2 < \sqrt{\frac{C}{A(C+D)}}$ we can guarantee that $\partial_2 H$ is transverse to both V^+ and V^-. We see that R_1, R_2 and R_3 can be chosen arbitrarily small, as long as we choose ϵ_2 small enough, so that we can arrange for $\partial_1 H$ to be an arbitrarily small neighborhood of K_1. □

4.3. **Preparing well-behaved knots for surgery.** In order to prove theorem 1.2 we will want to attach handles along well-behaved transverse knots, so now we present a method to turn well-behaved knots into knots which are prepared for surgery under certain conditions

Recall that, given a neighborhood ν of a knot K with contact structure ξ, if (r, μ, λ) is an almost normal coordinate system on ν then so is $(r, \mu - k\lambda, \lambda)$ for any $k \in \mathbb{Z}$. If $\xi = \ker \alpha^+$ and (α^+, α^-) is well-behaved with respect to (r, μ, λ) then (α^+, α^-) will also be well-behaved with respect to $(r, \mu - k\lambda, \lambda)$, and the structural data (A, B, C, D) with respect to (r, μ, λ) will change to $(A - Bk, B, C, D + Ck)$ with respect to $(r, \mu - k\lambda, \lambda)$. Thus, if we are willing to increase framings, it is easy to arrange that $D > 0$ and that $BD > 1$. However it is not clear how to arrange that $A = B$.

Lemma 4.7. *Suppose that (α^+, α^-) is well-behaved with respect to coordinates (r, μ, λ) on ν with structural data (A, B, C, D), where $D > 0$ and $BD > 1$. Then, for any $\epsilon > 0$, there exists a function $h : \nu \to [0, \infty)$ supported inside $\{r \leq \epsilon\}$ with the following properties: Let $S^+ = (\mathbb{R}, \omega^+)$ be the positive symplectification of $(\nu, (\alpha^+, \alpha^-))$, with dilation-contraction pair (V^+, V^-). Let $\nu_h = \{(h(p), p)\} \subset S^+$ and let $\pi : \nu_h \to \nu$ be the natural projection. Then ν_h is transverse to both vector fields, and the induced contact pair (α_h^+, α_h^-) is prepared for surgery with respect to the coordinates $(r \circ \pi, \mu \circ \pi, \lambda \circ \pi)$ on some (smaller) neighborhood of $\pi^{-1}(K)$.*

Proof. As mentioned earlier we may assume that $\alpha^+ = \frac{1}{B + Ar^2}(r^2 d\mu + d\lambda)$. To avoid too much notation, we will use (r, μ, λ) on ν_h to refer to $(r \circ \pi, \mu \circ \pi, \lambda \circ \pi)$. For a given h we will have $\alpha_h^+ = e^h \alpha^+$ on ν_h and lemma 4.3 gives us conditions for V^- to be transverse to ν_h. When V^- is also transverse to ν_h we get $\alpha_h^0 = \alpha^0$. Choose

a constant A_0 with $\frac{1}{D} < A_0 < B$ (we have $\frac{1}{D} < B$ because $BD > 1$ and $D > 0$). Note that we then have $A_0(C + D) > 1$ (because $C > 0$ and $D > 0$). We will construct h so that $\alpha_h^+ = \frac{1}{A_0+A_0r^2}(r^2 d\mu + d\lambda)$ on $\{r \leq \delta\} \subset \nu_h$ for some positive $\delta < \epsilon$; this together with $\alpha_h^0 = \alpha^0$ gives a contact pair on ν_h which is prepared for surgery on $\{r \leq \delta\}$.

Notice that this in fact determines h on $[0, \delta]$ because we must have $e^h = \frac{B+Ar^2}{A_0(1+r^2)}$ for $r \in [0, \delta]$. We should check that h so defined is in fact positive: $h \geq 0$ as long as $B + Ar^2 \geq A_0(1 + r^2)$, which will hold for small enough r as long as $B > A_0$, which is how we chose A_0. In other words, if we choose δ small enough we can guarantee that $h > 0$ on $\{r \leq \delta\}$.

Next we check that, with h thus defined on $\{r \leq \delta\}$, V^- is transverse to ν_h. Calculating and applying lemma 4.3 we see that transversality will hold if

$$e^h < AC + BD - \frac{(C - Dr^2)(B + Ar^2)}{2r}\frac{\partial h}{\partial r} \tag{4.4}$$

which, for our given h, becomes:

$$B + Ar^2 < A_0(D + C)(B + Ar^2)\ .$$

This holds for $r \leq \delta$ because $B + Ar^2 \geq A_0(1 + r^2)$ and $A_0(D + C) > 1$.

Now we should check that we can extend h to ν to have support inside $\{r \leq \epsilon'\}$ for some $\epsilon' < \epsilon$, in such a way that V^- remains transverse to ν_h. On $\{r \geq \epsilon'\}$ the transversality condition 4.4 above will be satisfied because h will be identically 0 and $AC + BD > 1$. For $r \leq \epsilon'$, if we choose ϵ' small enough we can replace condition 4.4 by the following simpler condition:

$$e^h < AC + BD - CB\frac{\partial h}{\partial r} \tag{4.5}$$

Using the facts that $AC + BD > 1$ and C and B are positive, it is easy to extend h to $r \leq \epsilon'$ maintaining this condition if $\frac{\partial h}{\partial r}(\delta) \leq 0$. If $\frac{\partial h}{\partial r}(\delta) > 0$ then, after perhaps making δ smaller still, we extend h to $\{r \leq \epsilon'\}$, with $h = 0$ near ϵ', in such a way that $\frac{\partial h}{\partial r}(r) < \frac{\partial h}{\partial r}(\delta)$ for all $r > \delta$ and that $\frac{\partial h}{\partial r}(r) < 0$ for all $r > \delta + \delta_1$, for some small $\delta_1 > 0$. This is enough to conclude that conditon 4.5 is met fir $r \leq \epsilon'$. □

Corollary 4.8. *Suppose that (X, ω) is a symplectic 4-manifold with ∂X partially convex and partially concave with induced contact pair (α^+, α^-), that $K \subset \partial X$ is a transverse knot with a neighborhood ν with coordinates (r, μ, λ) and that (α^+, α^-) is well-behaved with respect to (r, μ, λ) on ν with structural data (A, B, C, D). If $D > 0$ and $BD > 1$ then we can enlarge (X, ω) inside ν so as to arrange that ∂X is partially convex and partially concave with induced contact pair (α^+, α^-) which is prepared for surgery on ν with respect to (r, μ, λ). Then we can attach a handle as in proposition 4.6 along K with framing F_μ.*

Proof. Enlarge (X, ω) using the positive symplectification of $(\alpha^+|_\nu, \alpha^-|_\nu)$ (see lemma 4.1). Attach the subset $\{(t, p) \mid 0 \leq t \leq h(p)\}$ of this positive symplectification, where h comes from lemma 4.7, using the uniqueness of the symplectic germ $\mathcal{G}(\alpha^+|_\nu, \alpha^-|_\nu)$. □

5. From convexity to concavity via fibered links

In theorem 1.2, after attaching the handles, ∂Y comes with a link L', the union of the ascending circles of the handles. We will now prove theorem 1.2 and along

the way see the following characterization of the induced negative contact form α^-_Y on ∂Y.

Addendum 5.1 (to Theorem 1.2)**.** *There exists a closed tubular neighborhood τ of L, a constant k and a diffeomorphism ϕ from $\partial X \setminus \tau$ to $\partial_Y \setminus L'$ such that $kdp - \phi^*(\alpha^-_Y) = e^h\alpha$ for some function $h : \partial X \setminus \tau \to [0,\infty)$.*

Proof of Theorem 1.2 and addendum 5.1. We will first argue that we can enlarge (X,ω) so as to arrange that ∂X is in fact partially convex and partially concave, with induced contact pair (α^+,α^-), and so as to arrange that there are coordinates near each component of L, realizing the desired framing, with respect to which (α^+,α^-) is well-behaved satisfying the conditions of corollary 4.8.

Recall the notation in the definition of "nicely fibered". The transverse contact vector field is V and the fibration is $p : \partial X \setminus L \to S^1$. With (X,ω) as given, we have some induced contact form α on ∂X with $\xi = \ker\alpha$ such that $(\partial X, \xi, L, p)$ satisfy the definition of "nicely fibered". Consider a new contact form α^+ defined by $\alpha^+|_\xi = 0$ and $\alpha^+(V) = 1$. Notice that we then have $R_{\alpha^+} = V$. The new contact form α^+ has the same kernel as α, so $\alpha^+ = g\alpha$ for some function $g : \partial X \to \mathbb{R}\setminus\{0\}$. By replacing V with $-V$ if necessary we can arrange that $g > 0$. Also note that we can replace V with kV for any constant $k > 0$ without changing the "nicely fibered" condition, and thus we can arrange that $g > 1$ (using the compactness of ∂X). This means that we can enlarge (X,ω) so as to arrange that the induced contact form on ∂X is actually α^+, using the symplectification $(\mathbb{R}\times\partial X, d(e^t\alpha))$.

Now choose a constant c so that $c \cdot dp(V) > 1$ on $M \setminus L$ (this depends on the compactness of ∂X and on the fact that V and dp are invariant on a neighborhood of L so that $dp(V)$ is constant near L). Let $\alpha^0 = c \cdot dp$ and let $\alpha^- = \alpha^0 - \alpha^+$. Then (α^+,α^-) is a contact pair on ∂X with $-d\alpha^- = d\alpha^+$, so using corollary 4.4 we may now regard ∂X as partially concave and partially convex with induced contact pair (α^+,α^-). Of course α^+ is still defined on all of ∂X while α^- is only defined on $\partial X \setminus L$, and so for now α^- contains no new information. However, when we show that (α^+,α^-) is well-behaved near L we will be able to attach the handles from the previous section, after which α^- will extend across the new boundary and we will be able to forget about α^+ to conclude that the new boundary is concave. Thus α^- contains the seed of the concavity which we will achieve after attaching handles.

We see that (α^+,α^-) is well-behaved near L using the "nicely fibered" condition again. Near each component we know that there are normal coordinates (r,μ,λ) such that $\alpha^0(\partial_r) = 0$, $dr(V) = 0$ and such that V and α^0 are invariant under the flows of ∂_r, ∂_μ and ∂_λ. This immediately establishes that $\alpha^0 = Cd\mu + Dd\lambda$ for some constants C and D, and that $R_{\alpha^+} = A\partial_\mu + B\partial_\lambda$ for some constants A and B. Now we need to arrange that B and C are positive; this will follow from the orientation condition on the characteristic foliation on the fibers. Looking at our orientation convention, the fact that the foliation points radially inwards means that $\alpha^+ \wedge \alpha^0 \wedge (-dr) > 0$. Recall that we can take $\alpha^+ = \frac{1}{B+Ar^2}(r^2 d\mu + d\lambda)$ as our model, so we get that $\frac{C-Dr^2}{B+Ar^2} > 0$ for small r. Thus either B and C are both positive, or if they are both negative we can replace the coordinate system with $(r,-\mu,-\lambda)$ to get B and C both positive.

Now if we arrange that each coordinate system (r,μ,λ) for each component of L realizes the desired framing F of L, we see that the condition that F is positive with respect to the fibration p means exactly that, near each component, $-\frac{D}{C} < 0$ and since $C > 0$ this means that $D > 0$. Now we still may not have that $BD > 1$.

To arrange this we may need to again replace α^0 by $k\alpha^0$ for some constant $k > 1$ (again using compactness of ∂X), which will replace C and D with kC and kD. Now corollary 4.8 shows how to enlarge (X, ω) and attach handles. After enlarging (X, ω), we have ∂X partially convex and partially concave with induced contact pair (α^+, α^-), with $\partial^+ X = \partial X$ and $\partial^- X = \partial X \setminus L$. After attaching the handles ∂Y is partially convex and partially concave with induced contact pair (α_Y^+, α_Y^-) with domains $\partial^\pm Y$, with $\partial^+ Y = \partial Y \setminus L'$ and $\partial^- Y = \partial Y$ (where L' is the union of the ascending spheres). This means that we can ignore α_Y^+ and realize that in fact ∂Y is concave with induced negative contact form α_Y^-, and the characterization of α_Y^- in the addendum follows. □

5.1. **Examples.** First we will show that S^3 with surgery on either the unknot with any framing $F > 0$ or the Hopf link with any framing $F \geq 0$ can be realized as the concave boundary of a symplectic 4-manifold.

Let $(r_1, \theta_1, r_2, \theta_2)$ be polar coordinates on $\mathbb{R}^4$. Consider $S^3 = \partial B^4 \subset (\mathbb{R}^4, \omega)$ where $\omega = r_1 dr_1 \wedge d\theta_1 + r_2 dr_2 \wedge d\theta_2$. Then $V = \frac{1}{2}(r_1 \partial_{r_1} + r_2 \partial_{r_2})$ is a symplectic dilation transverse to S^3 inducing the standard positive contact form $\alpha = \frac{1}{2}(r_1^2 d\theta_1 + r_2^2 d\theta_2)$ on S^3. We compute that $R_\alpha = \partial_{\theta_1} + \partial_{\theta_2}$ and let $V = R_\alpha$, a transverse contact vector field. Now consider two cases:

1. $L = K = \{r_1 = 0\} \subset S^3$, the standard unknot. Consider the fibration $p = \theta_1 : S^3 \setminus L \to S^1$ and notice that $dp(V) > 0$. Polar coordinates near K are given by $(r = r_1, \mu = \theta_1, \lambda = \theta_2)$, from which we can verify that K is nicely fibered with fibration p and contact vector field V. With respect to these coordinates, F_μ is the standard 0-framing of K, and since $dp = d\mu$, the condition that a framing F is positive with respective to p is simply the condition that $F > 0$. Thus we can attach a single "convex-to-concave" handle along the unknot with any framing $F > 0$ to make a symplectic manifold with concave boundary.
2. $L = K_1 \cup K_2$ where $K_i = \{r_i = 0\} \subset S^3$, the standard Hopf Link. Now consider the fibration $p = \theta_1 + \theta_2 : S^3 \setminus L \to S^1$ and again, $dp(V) > 0$. Oriented polar coordinates near K_1 are as above and oriented polar coordinates near K_2 are given by $(r = r_2, \mu = \theta_2, \lambda = \theta_1)$ and again we verify that L is nicely fibered with fibration p and contact vector field V. Also F_μ is the standard 0-framing of each K_i. Now, however, $dp = d\mu + d\lambda$ near each K_i, so the condition that a framing F is positive with respect to p is the condition that the framing on each K_i is greater than -1. Thus we can attach a pair of "convex-to-concave" handles along the Hopf link as long as each handle is framed with framing 0 or larger, and the result is a symplectic manifold with concave boundary.

The first example generalizes. Given an n-punctured surface Σ with a proper Morse function $f_\Sigma : \Sigma \to [0, \infty)$ with only critical points of index 0 and 1, with the critical points of index 0 lying in $f^{-1}\{0\}$ and those of index 1 lying in $f^{-1}\{\frac{1}{4}\}$, we can use Weinstein's construction in dimension 2 to get a symplectic form ω_Σ on Σ and a gradient-like symplectic dilation V_Σ such that the structure is "standard" on, say, $f^{-1}(\frac{1}{2}, \infty)$. In other words, $f^{-1}(\frac{1}{2}, \infty)$ looks like n copies of $\mathbb{R}^2 \setminus \{r^2 \leq \frac{1}{2}\}$ with its standard symplectic form $r dr \wedge d\theta$, symplectic dilation $\frac{1}{2} r \partial_r$ and Morse function r^2. Consider the symplectic 4-manifold $(\Sigma \times \mathbb{R}^2, \omega = \omega_\Sigma + r dr \wedge d\theta)$ with the Morse function $f = f_\Sigma + r^2$. Note that $V = V_\Sigma + \frac{1}{2} r \partial_r$ is a gradient-like symplectic dilation

for f. Let $X = f^{-1}[0,1]$ and let $M = \partial X = f^{-1}(1)$; M is the convex boundary of (X,ω) with induced contact form $\alpha = \alpha_\Sigma + \frac{1}{2}r^2 d\theta$, where $\alpha_\Sigma = \imath_{V_\Sigma}\omega_\Sigma$. Note that M is diffeomorphic to the "boundary with smoothed corners" of the product of a disk and a compact surface of genus g with n boundary components. Alternately, if the handle decomposition of Σ is the usual one with one 0-handle and $(2g+n-1)$ 1-handles, then X is diffeomorphic to B^4 with $(2g+n-1)$ 1-handles attached, and M is diffeomorphic to S^3 with 0-surgery on $(2g+n-1)$ unlinked unknots.

We can decompose M into two open sets: $A = \{f_\Sigma < 1, r^2 = 1 - f_\Sigma\}$ and $B = \{0 \leq r^2 < \frac{1}{2}, \frac{1}{2} < f_\Sigma = 1 - r^2 \leq 1\}$. The set A is the complement of the n-component link $L = \{r^2 = 0, f_\Sigma = 1\}$, and the function $\theta : A = M \setminus L \to S^1$ is a fibration with fiber diffeomorphic to Σ. To see that $d\theta(R_\alpha) > 0$, note that this is equivalent to the requirement that $d\theta \wedge d\alpha > 0$. But $d\theta \wedge d\alpha = d\theta \wedge d\alpha_\Sigma = d\theta \wedge \omega_\Sigma > 0$. On B, the entire structure is identical to n copies of the structure described in the earlier example on S^3 in a neighborhood of the standard unknot. Thus we can attach handles along L with any framing larger than the "0-framing" determined by the fibration to create a concave symplectic 4-manifold.

To build on these constructions it is worth noting a simple variation of Weinstein's construction in [6]: Suppose (X,ω) is a symplectic 4-manifold with *concave* boundary and $K \subset \partial X$ is a Legendrian knot with respect to the induced negative contact structure on ∂X. Then a symplectic handle can be attached along K with framing $\mathrm{tb}(K) - 1$ such that the new symplectic manifold again has *concave* boundary. This can be seen as follows: Weinstein's "convex-to-convex" 4-dimensional 2-handles are constructed as subsets of $\mathbb{R}^4$ using the symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, the Morse function $f = -x_1^2 - x_2^2 + y_1^2 + y_2^2$ and the symplectic dilation $V = -x_1\partial_{x_1} + 2y_1\partial_{y_1} - x_2\partial_{x_2} + 2y_2\partial_{y_2}$. One checks that V is positively transverse to level sets of f and that the descending sphere $K_1 \subset f^{-1}\{\epsilon_1\}$ (for any $\epsilon_1 < 0$) is Legendrian with respect to the induced contact structure on $f^{-1}\{\epsilon_1\}$ and that the handle framing of K_1 is one less than the Thurston-Bennequin framing, so that such a handle can be attached along any Legendrian knot K with framing $\mathrm{tb}(K) - 1$. To construct "concave-to-concave" 2-handles, use the same symplectic form and Morse function but now consider the symplectic *contraction* $V = -2x_1\partial_{x_1} + y_1\partial_{y_1} - 2x_2\partial_{x_2} + y_2\partial_{y_2}$. Now simply check that this new V is positively transverse to level sets of f, that K_1 is Legendrian with respect to the induced negative contact structure on $f^{-1}\{\epsilon_1\}$ and that the handle framing is again equal to $\mathrm{tb}(K_1) - 1$.

In our example above ($X = f^{-1}[0,1] \subset \Sigma \times \mathbb{R}^2$ and $(M,\alpha) = (\partial X, \alpha_\Sigma + \frac{1}{2}r^2 d\theta)$), the characteristic foliation on each fiber Σ is given by the flow lines of the vector field V_Σ. We can construct Σ so that there will be some closed leaves of this singular foliation (containing singular points). Each such closed leaf is a Legendrian knot in (M,α) the Thurston-Bennequin framing of which is the framing given by the fiber. Let Y be the result of attaching 2-handles along $L = \{r^2 = 0, f_\Sigma = 1\}$, with ∂Y concave with induced negative contact form α_Y and fibered link L'. Notice that, under the diffeomorphism $\phi : \partial X \setminus \tau \to \partial Y \setminus L'$ of addendum 5.1, α_Y induces the same characteristic foliation on the fibers of θ as α did. Thus the closed leaves of the foliation are again Legendrian knots in $(\partial Y, \alpha_Y)$ and we can attach symplectic 2-handles along these knots (as in the previous paragraph) with framing -1 with respect to the fibers to build larger manifolds with concave boundary. This increases our class of examples of 3-manifolds which bound concave symplectic 4-manifolds; first perform any positive surgeries along the original n-component link L, then

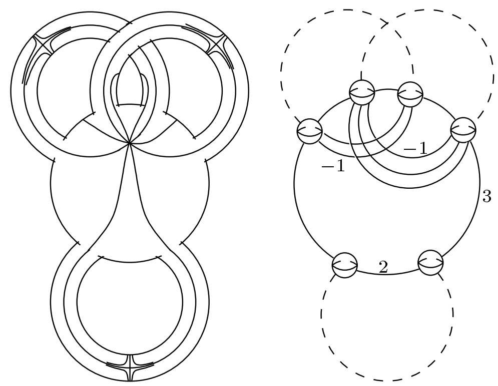

FIGURE 5. A handlebody decomposition of a surface with some leaves of the characteristic foliation giving a framed link description of a symplectic 4-manifold with concave boundary.

perform −1 surgeries on arbitrarily many copies of each closed leaf of the singular foliation (use different fibers to get the different copies). A random example of this construction is shown in figure 5.

These examples are fairly immediate and we hope that more sophisticated examples can be constructed using these techniques. The obvious challenge is to construct examples in which the negative contact structure on the concave boundary is recognizable as contactomorphic via an orientation reversing diffeomorphism to the positive contact structure on some other convex boundary, so that interesting closed symplectic manifolds can be constructed.

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